Année universitaire 2024/2025

Mathématiques Appliquées et Théoriques - 2e année de Master

Crédits ECTS : 60

LES OBJECTIFS DE LA FORMATION

Ce parcours est une restructuration de l'actuel parcours de M2 Analyse et Probabilités opéré par l'Université Paris Dauphine - PSL, dont un tiers des cours sont déjà proposés par les autres établissements de PSL. Le parcours Mathématiques Appliquées et Théoriques est une formation de pointe qui prépare les étudiants à un doctorat en mathématiques en leur offrant une formation théorique solide dans divers champs de mathématiques.

Les objectifs de la formation :

- Etre en capacité de modélisation et d'analyse
- Avoir un socle de connaissances solides en mathématiques appliquées et théoriques
- Aptitude à la recherche et à la recherche et développement
- Acquérir dans un large champ des mathématiques, et en particulier dans les domaines suivants : analyse des EDP, contrôle et analyse numérique ; probabilités et physique statistique ; géométrie et systèmes dynamiques.

PRÉ-REQUIS OBLIGATOIRES

Titulaires d'un diplôme BAC+4 (240 crédits ECTS) ou équivalent à Dauphine, d’une université ou d’un autre établissement de l’enseignement supérieur dans le domaine des mathématiques appliquées

POURSUITE D'ÉTUDES

Doctorat de mathématiques appliquées et théorique.
Ingénieur recherche et développement.

PROGRAMME DE LA FORMATION

- Semestre 3
  - Cours introductifs
    - A review of functional analysis tools for PDEs
    - A review of probability theory foundations
    - A review of differential calculus for ODEs and PDEs
  - Cours fondamentaux
    - Introduction to dynamical systems
    - Introduction to evolution PDEs
    - Introduction to non linear elliptic PDEs
    - Limit theorems and large deviations
    - Numerical methods for deterministic and stochastic problems
    - Stochastic calculus
  - Cours spécialisés
    - Continuous optimization

        - Course of PDE and applications
        - Differential geometry and gauge theory
        - Dynamics of semi-linear wave equation
        - Dynamics of gravitational systems with a large number of particles
        - Entropy methods, functional inequalities and applications
        - Gravitation classique et mécanique céleste
        - Integrable probability and the KPZ universality class
        - Introduction to control theory
        - Introduction to statistical mechanics and interacting particle systems
        - Lie Groups, Lie algebras and represention s
        - Mean field games theory
        - Mixing times of Markov chains
        - Numerical methods for fluid dynamics
        - Pathwise (rough) stochastic analysis
        - Random geometric models
        - Random walks and random media
        - Spectral theory and variational methods
        - Stochastic control
        - Variational and geodesic methods for Image Analysis
        - Variational problems and optimal transport
- Semestre 4
    - Bloc mémoire
        - Mémoire de recherche

DESCRIPTION DE CHAQUE ENSEIGNEMENT

A review of differential calculus for ODEs and PDEs

ECTS : 0

Volume horaire : 15

Description du contenu de l'enseignement :

We will revise the main notions and theorems from differential calculus (implicit function theorem, inverse function theorem, Brouwer theorem...), as well as main facts about ODE and results about linear and nonlinear stability and smooth dependance by perturbations.

A review of functional analysis tools for PDEs

ECTS : 0

Volume horaire : 15

Description du contenu de l'enseignement :

- Lp spaces, Sobolev spaces
- Distributions, Fourier transform, Laplace, heat and Schrödinger equations in the whole space
- Self-adjoint compact operators
- Laplace and Poisson equations in a domain.

A review of probability theory foundations

ECTS : 0

Volume horaire : 15

Description du contenu de l'enseignement :

- Random variables, expectations, laws, independence
- Inequalities and limit theorems, uniform integrability
- Conditioning, Gaussian random vectors

- Bounded variation and Lebegue-Stieltjes integral
- Stochastic processes, stopping times, martingales
- Brownian motion: martingales, trajectories, construction
- Wiener stochastic integral and Cameron-Martin formula.

Continuous optimization

ECTS : 6

Volume horaire : 24

Description du contenu de l'enseignement :

This course will cover the bases of continuous, mostly convex optimization. Optimization is an important branch of applied industrial mathematics. The course will mostly focus on the recent development of optimization for large scale problems such as in data science and machine learning. A first part will be devoted to setting the theoretical grounds of convex optimization (convex analysis, duality, optimality conditions, non-smooth analysis, iterative algorithms). Then, we will focus on the improvement of basic first order methods (gradient descent), introducing operator splitting, acceleration techniques, non-linear ("mirror") descent methods and (elementary) stochastic algorithms.

Course of PDE and applications

ECTS : 6

Volume horaire : 18

Differential geometry and gauge theory

ECTS : 6

Dynamics of gravitational systems with a large number of particles

ECTS : 6

Volume horaire : 21

Description du contenu de l'enseignement :

Reminder on differential equations

- Reminder on Hamiltonian systems
- Short reminder on measure and integration theories
- Elements on distributions
- Application to the Vlasov equation
- The Vlasov-Poisson system
- The BBGKY hierarchy, the hypothesis of molecular chaos
- The particular case of a cluster with spherical symmetry, an explicit solution

This course is taught at Observatoire de Paris.

Bibliographie, lectures recommandées :

- Binney - Tremaine : Galactic Dynamics
- Cours polycopiés de F. Golse

Dynamics of semi-linear wave equation

ECTS : 6

Volume horaire : 28

Description du contenu de l'enseignement :

The aim of this course is to present recent developments concerning the dynamics of non-linear wave equations.

In the first part of the course, I will present some classical properties of linear wave equations (cf. [3, Chapter 5]): representation of solutions, finite speed of propagation, asymptotic behavior, dispersion and Strichartz inequalities [7, 5].

The second part of the course concerns semi-linear wave equations. After a presentation of the basic properties of these equations (local existence and uniqueness of solutions, conservation laws, transformations cf. e.g. [5, 6]), I'll give several examples of dynamics: scattering to a linear solution, self-similar behavior and solitary waves. I will also give results on the classification of the dynamics for the energy critical wave equation following [2, 4], and some elements of proofs, including the profile decomposition introduced by Bahouri and Gérard [1].

The prerequisites are the basics of classical real and harmonic analysis. This course can be seen as a continuation of the fundamental courses *Introduction to Nonlinear Partial Differential Equations* and *Introduction to Evolutionary Partial Differential Equations*, but can also be taken independently of these two courses.

This course will be taught at ENS.

Bibliographie, lectures recommandées :

1. Bahouri, H., and Gérard, P. High frequency approximation of solutions to critical nonlinear wave equations. Amer. J. Math. 121, 1 (1999), 131–175.
2. Duyckaerts, T., Kenig, C., and Merle, F. Classification of radial solutions of the focusing, energy-critical wave equation. Camb. J. Math. 1, 1 (2013), 75–144.
3. Folland, G. B. Introduction to partial differential equations., 2nd ed. ed. Princeton, NJ: Princeton University Press, 1995.
4. Kenig, C. E. Lectures on the energy critical nonlinear wave equation, vol. 122 of CBMS Reg. Conf. Ser. Math. Providence, RI: American Mathematical Society (AMS), 2015.
5. Sogge, C. D. Lectures on nonlinear wave equations. Monographs in Analysis, II. International Press, Boston, MA, 1995.
6. Strauss, W. A. Nonlinear wave equations, vol. 73 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1989.
7. Tao, T. Nonlinear dispersive equations, vol. 106 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. Local and global analysis.

Entropy methods, functional inequalities and applications

ECTS : 6

Volume horaire : 24

Description du contenu de l'enseignement :

Various functional inequalities are classically seen from a variational point of view in nonlinear analysis. They also have important consequences for evolution problems. For instance, entropy estimates are standard tools for relating rates of convergence towards asymptotic regimes in time-dependent equations with optimal constants of various functional inequalities. This point of view applies to linear diffusionsand will be illustrated by some results on the Fokker-Planck equation based on the "carré du champ" method introduced by D. Bakry and M. Emery. In the recent years,the method has been extended from linear to nonlinear diffusions. This aspect will be illustrated by results on Gagliardo-Nirenberg-Sobolev inequalities on the sphere and on the Euclidean space. Even the evolution equations can be used as a tool for the study of detailed properties of optimal functions in inequalities and their refinements. There are also applications to other equations than pure diffusions: hypocoercivity in kinetic equations is one of them. In any case, the notion of entropy has deep roots in statistical mechanics, with applications in various areas of science ranging from mathematical physics to models in biology. A special emphasis will be put during the course on the corresponding models which offer many directions for new research development.

Gravitation classique et mécanique céleste

ECTS : 6

Volume horaire : 30

Description du contenu de l'enseignement :

This course is taught in French at Observatoire de Paris.
La mécanique céleste est plus vivante que jamais. Après un renouveau résultant de la conquête spatiale et de la nécessité des calculs des trajectoires des engins spatiaux, un deuxième souffle est apparu avec l'étude des phénomènes chaotiques. Cette dynamique complexe permet des variations importantes des orbites des corps célestes, avec des conséquences physiques importantes qu'il faut prendre en compte dans la formation et l'évolution du système solaire. Avec la découverte des planètes extra solaires, la mécanique céleste prend un nouvel essor, car des configurations qui pouvaient paraître académiques auparavant s'observent maintenant, tellement la diversité des systèmes observés est grande. La mécanique céleste apparaît aussi comme un élément essentiel permettant la découverte et la caractérisation des systèmes planétaires qui ne sont le plus souvent observés que de manière indirecte.
Le cours a pour but de fournir les outils de base qui permettront de mieux comprendre les interactions dynamiques dans les systèmes gravitationnels, avec un accent sur les systèmes planétaires, et en particulier les systèmes planétaires extra solaires.

Le cours vise aussi à présenter les outils les plus efficaces pour la mise en forme analytique et numérique des problèmes généraux des systèmes dynamiques conservatifs.
Plan:

- Le problème des deux corps. Aperçu de quelques intégrales premières, réduction du nombre de degrés de liberté, trajectoire, évolution temporelle. Développements classiques du problème des deux corps
- Introduction à la mécanique analytique. Principe de moindre action, Lagrangien, Hamiltonien
- Équations canoniques. Crochets de Poisson, intégrales premières, transformations canoniques
- Propriétés des systèmes Hamiltoniens. Systèmes intégrables. Flot d'un système Hamiltonien
- Intégrateurs numériques symplectiques
- Systèmes proches d'intégrable. Perturbations. Série de Lie
- Développement du potentiel en polynômes de Legendre
- Évolution à long terme d'un système planétaire hiérarchique, mécanisme de Lidov- Kozai. Application aux exoplanètes
- Mouvements chaotiques
- Exposants de Lyapounov
- Analyse en fréquence.

Integrable probability and the KPZ universality class

ECTS : 6

Volume horaire : 24

Description du contenu de l'enseignement :

Integrable probability is a relatively new subfield of probability that concerns the study of exactly solvable probabilistic models and their underlying algebraic structures. Most of these so-called integrable models come from statistical physics. They serve as toy models to discover the asymptotic behavior common to large classes of models, called universality classes. The methods used in integrable probability often come from other areas of mathematics (such as representation theory or algebraic combinatorics) and from theoretical physics. In the last twenty years, these methods have been particularly fruitful for studying the Kardar-Parisi-Zhang universality class (named after the three physicists who pioneered the domain in the 1980s). This class gathers interface growth models describing a wide variety of physical phenomena, whose asymptotic behavior is surprisingly related to the theory of random matrices.
This course will focus on a central tool in the eld: Schur and Macdonald processes. This will allow us to study in a uni ed way some of the most emblematic integrable models, and ultimately arrive at the the exact calculation of the law of a solution of the Kardar-Parisi-Zhang equation. Along the way, we will take a few detours through various applications or related concepts: random matrices, Robinson-Schensted-Knuth correspondence, interacting particle systems, Yang-Baxter equation and the six-vertex model, random walks in a random environment.
The course will be taught at ENS.

Introduction to control theory

ECTS : 6

Volume horaire : 28

Description du contenu de l'enseignement :

This course focuses on an introduction to systems and control theory. It concerns the study of a dynamical system affected by an input signal which we aim at designing to modify the system behavior. It will focus on nonlinear Ordinary Differential Equations (ODEs), but will also include an introduction to the control of Partial Differential Equations.
We will start by reviewing stability notions of nonlinear ODEs (Lyapunov theorems, sufficient and necessary stability conditions, spectral criteria for linear systems, Input-to-State Stability,…). Then, we will study the concepts of controllability/observability of dynamical systems and move to stabilization of equilibrium points, with the presentation of a few control design methodologies (backstepping, forwarding, optimal control, Lie Bracket methods...).
The class will be concluded by a few session on the extension of these concepts to infinite-dimensional linear control systems, namely, Partial Differential Equations. Examples will include in-domain and/or boundary control of the heat equation and the wave equation.
The course will be taught at École des Mines.

Introduction to dynamical systems

ECTS : 6

Volume horaire : 30

Description du contenu de l'enseignement :

1. Examples of dynamical systems in discrete and continuous time (circle rotation, shift, hyperbolic dynamical system, horseshoe, flow, section and suspension, attractor)
2. Topological dynamics, circle homeomorphisms and Poincaré classification, hyperbolic dynamics (geodesic flow, horocyclic flow)

Bibliographie, lectures recommandées :

- V.I. Arnold, *Ordinary differential equations* (contains prerequisite matters)
- V.I. Arnold, *Geometric methods in the theory or ordinary differential equations* (further reading)
- M. Brin and G. Stuck, *Introduction to dynamical systems* (great introduction to the field)

Introduction to evolution PDEs

ECTS : 6

Volume horaire : 37.5

Description du contenu de l'enseignement :

In a first part, we will present several results about the well-posedness issue for evolution PDE.

- Parabolic equation. Existence of solutions for parabolic equations by the mean of the variational approach and the existence theorem of J.-L. Lions.
- Transport equation. Existence of solutions by the mean of the characterics method and renormalization theory of DiPerna-Lions. Uniqueness of solutions thanks to Gronwall argument and duality argument.
- Evolution equation and semigroup. Linear evolution equation, semigroup and generator. Duhamel formula and mild solution. Duality argument and the well-posedness issue. Semigroup Hille-Yosida-Lumer-Phillips' existence theory.

In a second part, we will mainly consider the long term asymptotic issue.

- More about the heat equation. Smoothing effect thanks to Nash argument. Rescaled (self-similar) variables and Fokker-Planck equation. Poincaré inequality and long time asymptotic (with rate) in L2 Fisher information, log Sobolev inequality and long time convergence to the equilibrium (with rate) in L1.
- Entropy and applications. Dynamical system, equilibrium and entropy methods. Self-adjoint operator with compact resolvent. A Krein-Rutman theorem for conservative operator. Relative entropy for linear and positive PDE. Application to a general Fokker-Planck equation. Weighted L2 inequality for the scattering equation.
- Markov semigroups and the Harris-Meyn-Tweedie theory.

In a last part, we will investigate how the different tools we have introduced before can be useful when considering a nonlinear evolution problem.

- The parabolic-elliptic Keller-Segel equation. Existence, mass conservation and blow up. Uniqueness. Self-similarity and long time behavior.

Introduction to non linear elliptic PDEs

ECTS : 6

Volume horaire : 37.5

Description du contenu de l'enseignement :

- Existence of weak solutions of linear and nonlinear elliptic PDEs by variational methods
- Regularity of weak solutions to linear and nonlinear elliptic PDEs
- Maximum principles and applications
- Brouwer degree, Leray-Schauder degree, fixed-point theorems
- Local and global bifurcation theory applied to nonlinear elliptic PDEs

Bibliographie, lectures recommandées :

- L.C. Evans, *Partial Differential equations* (Graduate Studies in Mathematics 19, AMS).
- L. Nirenberg, *Topics in Nonlinear Functional Analysis* (Courant Lecture Notes Series 6, AMS).

Introduction to statistical mechanics and interacting particle systems

ECTS : 6

Volume horaire : 30

Description du contenu de l'enseignement :

The aim of statistical mechanics is to understand the macroscopic behavior of a physical system by using a probabilistic model containing the information for the microscopic interactions. The goal of this course is to give an introduction to this broad subject, which lies at the intersection of many areas of mathematics: probability, graph theory, combinatorics, algebraic geometry...
In the first part of the course we will introduce the key notions of equilibrium statistical mechanics. In particular we will study the phase diagram of the following models: Ising model (ferromagnetism), dimer models (crystal surfaces) and percolation (flow of liquids in porous materials). In the second part we will introduce interacting particle systems, a large class of Markov processes used to model dynamical phenomena arising in physics (e.g. the kinetically constrained models for glasses) as well as in other disciplines such as biology (e.g. the contact model for the spread of infections) and social sciences (e.g. the voter model for the dynamics of opinions).

Lie Groups, Lie algebras and representions

ECTS : 6

Description du contenu de l'enseignement :

The theory of groups and their representations is a central topic which studies symmetries in various contexts occurring in pure or applied mathematics as well as in other sciences, most notably in physics.
Lie theory (i.e. the study of Lie groups and Lie algebras) has played an important role in mathematic ever since its introduction by the Norwegian mathematician Sophus Lie in the 19th century. It has had a profound impact in physics as well.
The aim of this course is to provide an introduction, from the mathematical perspective, of the classical concepts and techniques of Lie theory. The course will in particular deal with Lie groups, Lie algebras (of finite dimension) and their representations, and include the study of numerous examples.
This course will be taught at ENS.
Link to the course

Limit theorems and large deviations

ECTS : 6

Volume horaire : 30

Description du contenu de l'enseignement :

The first part of the course (5*3 hours) is devoted to the study of convergence of probability measures on general (that is not necessarily R or Rn) metric spaces or, equivalently, to the convergence in law of random variables taking values in general metric spaces. If this study has its own interest it is also useful to prove convergence of sequences of random objects in various random models that appear in probability theory. The main example we have to keep in mind is Donsker theorem that states that the path of a simple random walk on Z converges after proper renormalization to a brownian motion. We will start this course with some properties of probability measures on metric spaces and in particular on C([0, 1]), the space of real continuous function on [0, 1]. We will then study convergence of probability measures, having for aim Prohorov theorem that provides a useful characterization of relative compatctness via tightness. Finally we will gather everything to study convergence in law on C([0, 1]) and prove Donsker therorem. If there is still time we will consider other examples of applicatioin. The main reference for this first part of the course is Convergence of probability measures, P. Billingsley (second edition).

The second part of the course will deal with the theory of large deviations. This theory is concerned with the exponential decay of large fluctuations in random systems. We will try to focus evenly on establishing rigorours results and on discussing applications. First, we will introduce the basic notions and theorems: the large deviation principle, Kramer theorem for independent variables, as well as GŁrtner-Ellis and Sanovs theorems. Next, we will see some applications of the formalism. The examples are mainly inspired by equilibrium statistical physics and thermodynamics. They include the equivalence of ensembles, the interpretation of thermodynamical potentials as large deviation functionals, and phase transitions in the mean-field Curie-Weiss model. In a third part, we will develop large deviation principles for Markovian dynamical processes. If times allows, we will present some applications of these results in a last part of the course. There is no explicit prerequisite to follow the classes but students should be well acquainted with probability theory.

Mean field games theory

ECTS : 6

Volume horaire : 18

Description du contenu de l'enseignement :

The course on Stochastic Control (1rst semester) is a necessary prerequisite.
Mean field games is a new theory developed by Jean-Michel Lasry and Pierre-Louis Lions that is interested in the limit when the number of players tends towards infinity in stochastic differential games. This gives rise to new systems of partial differential equations coupling a Hamilton-Jacobi equation (backward) to a Fokker-Planck equation (forward). We will present in this course some results of existence, uniqueness and the connections with optimal control, mass transport and the notion of partial differential equations on the space of probability measures.

Mixing times of Markov chains

ECTS : 6

Volume horaire : 24

Description du contenu de l'enseignement :

How many times must one shuffle a deck of 52 cards? This course is a self-contained introduction to the modern theory of mixing times of Markov chains. It consists of a guided tour through the various methods for estimating mixing times, including couplings, spectral analysis, discrete geometry, and functional inequalities. Each of those tools is illustrated on a variety of examples from different contexts: interacting particle systems, card shufflings, random walks on groups, graphs and networks, etc. Finally, a particular attention is devoted to the celebrated cutoff phenomenon, a remarkable but still mysterious phase transition in the convergence to equilibrium of certain Markov chains.
Further information: www.ceremade.dauphine.fr/~salez/mix.html

Bibliographie, lectures recommandées :

- Notes de cours, examen 2019 et correction (J. Salez)
- Markov Chains and Mixing Times (D. Levin, Y. Peres & E. Wilmer)
- Mathematical Aspects of Mixing Times in Markov Chains (R. Montenegro & P. Tetali)
- Mixing Times of Markov Chains: Techniques and Examples (N. Berestycki)
- Reversible Markov Chains and Random Walks on Graphs (D. Aldous & J. Fill)

Mémoire de recherche

ECTS : 18

Numerical methods for deterministic and stochastic problems

ECTS : 6

Volume horaire : 45

Description du contenu de l'enseignement :

This course is an introduction to methods for the numerical solution of deterministic and stochastic differential equations and numerical aspects of machine learning. It consists of three distincts parts and includes implementations using Python, FreeFEM++ and Keras/Tensorflow.
Part 1: numerical methods for deterministic partial differential equations

- finite difference methods
- finite element methods
- spectral methods
- review of numerical methods for ordinary differential equations

Part 2: Monte Carlo methods for particle transport

- Monte Carlo integration
- convergence and variance reduction
- transport equations starting from probability measures: examples and numerical methods including particle methods

Part 3: machine learning and numerical statistics

- high-dimensional statistics and machine learning
- stochastic optimization : SGD, Adam, RMSProp, etc.
- neural networks: architecture, generative paradigms (VAE, GANs, "stable diffusion")

Bibliographie, lectures recommandées :

Part 1

- Randall J. LeVeque, "Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-dependent Problems", SIAM (2007)
- Alexandre Ern, Jean-Luc Guermond, "Theory and Practice of Finite Elements", Springer (2004)
- Jie Shen, Tao Tang, Li-Lian Wang, "Spectral Methods. Algorithms, Analysis and Applications", Springer (2011)

Part 2

- C. Graham, D. Talay, "Stochastic Simulation and Monte Carlo Methods", Springer (2013)
- B. Lapeyre, E. Pardoux, R. Sentis, "Introduction to Monte-Carlo Methods for Transport and Diffusion Equations", OUP Oxford (2003)

Part 3

- Ian Goodfellow, Yoshua Bengio, Aaron Courville, "Deep Learning", The MIT Press (2016)
- Alain Berlinet, Christine Thomas-Agnan "Reproducing Kernel Hilbert Spaces in Probability and Statistics", Springer (2011)

See also G. Turinici's web site.

Numerical methods for fluid dynamics

ECTS : 6

Volume horaire : 21

Pathwise (rough) stochastic analysis

ECTS : 6

Volume horaire : 24

Description du contenu de l'enseignement :

The pathwise (or rough) approach to stochastic analysis consists in a series of analytic methods, developed in the last two decades, which are able to deal with the functions of low regularity which arise naturally when considering stochastic objects. The crucial step is typically to identify a natural metric space on which the "noise-to-solution" map is continuous.
These methods allow on the one hand for a new (and often more natural and robust) point of view on classical objects (such as the SDE defined by Itô integration), while also allowing to deal with objects which were previously out of reach (such as ODE driven by fractional Brownian motion, or singular stochastic PDE).
The class will consist in an overview of these various techniques, with an emphasis on the simple case of Lyons' rough path theory. It will be mostly self-contained (a basic knowledge of stochastic calculus will be helpful).

Outline

1. Sewing lemma and Young integration
2. Basics of rough path theory (Rough path spaces, Rough integration and differential equations)
3. Application to stochastic equations (Brownian motion as a rough path, extensions to Gaussian processes)
4. Stochastic sewing (and applications to regularization by noise of ODE)
5. Introduction to singular SPDE (regularity structures or paracontrolled distributions)

The course will be taught at ENS.

Bibliographie, lectures recommandées :

P. Friz and M. Hairer, *A course on rough paths*

Random geometric models

ECTS : 6

Volume horaire : 26

Description du contenu de l'enseignement :

This course provides a quick access to some popular models in the theory of random graphs, point processes and random sets. These models are widely used for the mathematical analysis of networks that arise in different applications: communication and social networks, transportation, biology... We will discuss among the others: the Erdos-Reny graph, the configuration model, unimodular random graphs, Poisson point processes, hard core point processes, continuum percolation, Boolean model and coverage process, and stationary Voronoi percolation. Our main goal will be to discuss the similarities and the fundamental relationships between the different models.

Random walks and random media

ECTS : 6

Volume horaire : 30

Description du contenu de l'enseignement :

- **Random walks in random environment** are random processes obtained after launching a Markovian walker on **Z**d equipped with a random field of transition probabilities. We will review classical results (recurrence / transience, LLN, Sinai regime, Kesten Kozlov Spitzer regime) in dimension d=1 where the behaviour of the walk is well understood but also study the difficult multidimensional case d>=2 where even simple questions (as LLN) remains open.
- **Potential theory and electrical networks**: the analogy with electrical networks gives a physical insight as well as a robust method for proving recurrence or transience of reversible random walks on the Euclidean lattice or more general graphs.
- **Random interlacement**, introduced by Sznitman in the early 2010, may be seen as a « soup » of random walk paths. It plays an decisive role both as a limit object for many random walk models and also as a tractable long range correlated random field.

Spectral theory and variational methods

ECTS : 6

Volume horaire : 20

Stochastic calculus

ECTS : 6

Volume horaire : 45

Description du contenu de l'enseignement :

The first part of the course presents stochastic calculus for continuous semi-martingales. The second part of the course is devoted to Brownian stochastic differential equations and their links with partial differential equations. This course is naturally followed by the course "Jump processes".

- Probability basics
- Stochastic processes
- Brownian motion, Continuous semi-martingales, Stochastic integral, Itô's formula for semi-martingales and Girsanov's theorem Stochastic differential equations
- Diffusion processes Feynman-Kac formula and link with the heat equation Probabilistic representation of the Dirichlet problem

Stochastic control

ECTS : 6

Volume horaire : 24

Description du contenu de l'enseignement :

PDEs and stochastic control problems naturally arise in risk control, option pricing, calibration, portfolio management, optimal book liquidation, etc. The aim of this course is to study the associated techniques, in particular to present the notion of viscosity solutions for PDEs.

- Relationship between conditional expectations and parabolic linear PDEs
- Formulation of standard stochastic control problems: dynamic programming principle.
- Hamilton-Jacobi-Bellman equation
- Verification approach Viscosity solutions (definitions, existence, comparison)
- Application to portfolio management, optimal shutdown and switching problems

Variational and geodesic methods for Image Analysis

ECTS : 6

Volume horaire : 24

Description du contenu de l'enseignement :

This course, after giving a short introduction to digital image processing, will present an overview of variational methods for Image segmentation. This will include deformable models, known as active contours, solved using finite differences, finite elements, level sets method, fast marching method. A large part of the course will be devoted to geodesic methods, where a contour is found as a shortest path between two points according to a relevant metric. This can be solved efficiently by fast marching methods for numerical solution of the Eikonal equation. We will present cases with metrics of different types (isotropic, anisotropic, Finsler) in different spaces. All the methods will be illustrated by various concrete applications, like in biomedical image applications.

Variational problems and optimal transport

ECTS : 6

Volume horaire : 24

Description du contenu de l'enseignement :

Chapter 1: Convexity in the calculus of variations

- separation theorems, Legendre transforms, subdifferentiability
- convex duality by a general perturbation argument, special cases (Fenchel-Rockafellar, linear programming, zero sum games, Lagrangian duality)
- calculus of variations: the role of convexity, relaxation, Euler-Lagrange equations

Chapter 2: The optimal transport problem of Monge and Kantorovich

- The formulations of Monge and Kantorovich, examples and special cases (dimension one, the assignment problem, Birkhoff theorem), Kantorovich as a relaxation of Monge
- Kantorovich duality
- Twisted costs, existence of Monge solutions, Brenier’s theorem, Monge-Ampère equation, OT proof of the isoperimetric inequality
- the distance cost case and its connection with minimal flows

Chapter 3: Dynamic optimal transport, Wasserstein spaces, gradient flows

- Wasserstein spaces
- Benamou-Brenier formula and geodesics, displacement convexity
- gradient flows, a starter: the Fokker-Planck equation, general theory for lambda-convex functionals

Chapter 4: Computational OT and applications

- Entropic OT, Sinkhorn algorithm and its convergence
- Matching problems, barycenters,
- Wasserstein distances as a loss, Wasserstein GANs
